

# Scalar effective action in Krein space quantization

A. Refaei<sup>1,2,\*</sup>, M.V. Takook<sup>1†</sup>

September 14, 2011

<sup>1</sup>*Department of Physics, Razi University, Kermanshah, Iran*

<sup>2</sup>*Islamic Azad University, Sanandaj branch, Sanandaj, Iran.*

## Abstract

In this paper, the  $\lambda\phi^4$  scalar field effective action, in the one-loop approximation, is calculated by using the Krein space quantization. We show that the effective action is naturally finite and the singularity does not appear in the theory. The physical interaction mass, the running coupling constant and  $\beta$ -function are then calculated. The effective potential which is calculated in the Krein space quantization is different from the usual Hilbert space calculation, however we show that  $\beta$ -function is the same in the two different methods.

*PACS numbers:* 04.62.+v, 03.70+k, 11.10.Cd, 98.80.H

## 1 Introduction

The quantum gravity is one of the most important problem in theoretical physics. The linear quantum gravity in the background field method is perturbatively non-renormalizable and also there appear an infrared divergence. This infrared divergence does not manifest itself in the quadratic part of the effective action in the one-loop approximation. This means that the pathological behavior of the graviton propagator may be gauge dependent and so should not appear in an effective way as a physical quantity [1]. The infrared divergence which appears in the linear gravity in de Sitter space is the same as the minimally coupled scalar field in de Sitter space [2, 3]. It is shown that one can not construct a covariant quantization of the minimally coupled scalar field with only positive norm states [4]. It has been proved that the use of the two sets of solutions (positive and

---

\*e-mail: abr412@gmail.com

†e-mail: takook@razi.ac.ir

negative norms states) is an unavoidable feature if one wants to preserve causality (locality), covariance and elimination of the infrared divergence in quantum field theory for the minimally coupled scalar field in de Sitter space [5, 6], *i.e.* Krein space quantization.

The singular behavior of Green function at short relative distances (ultraviolet divergence) or in the large relative distances (infrared divergence) leads to main divergences in the quantum field theory. It was conjectured that quantum metric fluctuations might smear out the singularities of Green functions on the light cone, but it does not remove other ultraviolet divergences [8]. However, it has been shown that quantization in Krein space removes all ultraviolet divergences of quantum field theory (QFT) except the light cone singularity [7]. By using the Krein space quantization and the quantum metric fluctuations in the linear approximation, we showed that the infinities in the Green function are disappeared [8, 9].

Quantization in Krein space instead of Hilbert space has some interesting features. For example in this method, the vacuum energy becomes zero naturally, so the normal ordering would not be necessary [5, 7]. The auxiliary negative norm states, which are used in the Krein space quantization, play the regularization of the theory.

In the present work, using the Krein space method and quantum metric fluctuation at the linear approximation, we calculate the one loop effective action for scalar field. It has been shown that this effective action is naturally regularized. This effective action differs with what previously reported [10], however the  $\beta$ -functions are exactly the same. The effective mass and coupling constant are also calculated in this method. In this approximation, we see that this quantization eliminates the singularity in the theory without changing the  $\beta$ -functions. In the appendix, details of our calculation have been presented.

## 2 Scalar Green function

In this section, we review the elementary facts about Krein space quantization. A classical scalar field  $\phi(x)$  satisfies the following field equation

$$(\square + m^2)\phi(x) = 0 = (\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi(x), \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (2.1)$$

Inner or *Klein-Gordon* product and related norm are defined by [11]

$$(\phi_1, \phi_2) = -i \int_{t=\text{const.}} \phi_1(x) \overleftrightarrow{\partial}_t \phi_2^*(x) d^3x. \quad (2.2)$$

Two sets of solutions are given by:

$$u_p(k, x) = \frac{e^{i\vec{k}\cdot\vec{x}-iwt}}{\sqrt{(2\pi)^3 2w}} = \frac{e^{-ik.x}}{\sqrt{(2\pi)^3 2w}}, \quad u_n(k, x) = \frac{e^{-i\vec{k}\cdot\vec{x}+iwt}}{\sqrt{(2\pi)^3 2w}} = \frac{e^{ik.x}}{\sqrt{(2\pi)^3 2w}}, \quad (2.3)$$

where  $w(\vec{k}) = k^0 = (\vec{k} \cdot \vec{k} + m^2)^{\frac{1}{2}} \geq 0$ , note that  $u_n$  has the negative norm. In Krein space the quantum field is defined as follows [7]

$$\phi(x) = \frac{1}{\sqrt{2}}[\phi_p(x) + \phi_n(x)], \quad (2.4)$$

where

$$\begin{aligned} \phi_p(x) &= \int d^3\vec{k} [a(\vec{k})u_p(k, x) + a^\dagger(\vec{k})u_p^*(k, x)], \\ \phi_n(x) &= \int d^3\vec{k} [b(\vec{k})u_n(k, x) + b^\dagger(\vec{k})u_n^*(k, x)]. \end{aligned}$$

$a(\vec{k})$  and  $b(\vec{k})$  are two independent operators. The time-ordered product propagator for this field operator is

$$iG_T(x, x') = \langle 0 | T\phi(x)\phi(x') | 0 \rangle = \theta(t - t')\mathcal{W}(x, x') + \theta(t' - t)\mathcal{W}(x', x). \quad (2.5)$$

In this case we obtain

$$G_T(x, x') = \frac{1}{2}[G_F(x, x') + (G_F(x, x'))^*] = \Re G_F(x, x'), \quad (2.6)$$

where the Feynman Green function is defined by [11]

$$\begin{aligned} G_F(x, x') &= \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-x')} \tilde{G}_F(p) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x-x')}}{p^2 - m^2 + i\epsilon} \\ &= -\frac{1}{8\pi} \delta(\sigma_0) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0}) - iN_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}} - \frac{im^2}{4\pi^2} \theta(-\sigma_0) \frac{K_1(\sqrt{-2m^2\sigma_0})}{\sqrt{-2m^2\sigma_0}}, \end{aligned} \quad (2.7)$$

in which  $\sigma_0 = \frac{1}{2}(x - x')^2$ . So we have

$$G_T(x, x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-x')} \mathcal{P}\mathcal{P} \frac{1}{p^2 - m^2} = -\frac{1}{8\pi} \delta(\sigma_0) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}, \quad x \neq x', \quad (2.8)$$

$\mathcal{P}\mathcal{P}$  stands for the principal parts. Contribution of the coincident point singularity ( $x = x'$ ) merely appears in the imaginary part of  $G_F$  ([6] and equation (9.52) in [11])

$$G_F(x, x) = -\frac{2i}{(4\pi)^2} \frac{m^2}{d-4} + G_F^{\text{finit}}(x, x),$$

where  $d$  is the space-time dimension and  $G_F^{\text{finit}}(x, x)$  becomes finite as  $d \rightarrow 4$ . Note that the singularity of the Eq. (2.8) takes place only on the cone *i.e.*,  $x \neq x'$ ,  $\sigma_0 = 0$ .

It has been shown that the quantum metric fluctuations remove the singularities of Green's functions on the light cone [8]. Therefore, the quantum field theory in Krein space,

including the quantum metric fluctuation ( $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ), removes all the ultraviolet divergencies of the theory [9, 8], so one can write:

$$\langle G_T(x - x') \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle\sigma_1^2\rangle}} \exp\left(-\frac{\sigma_0^2}{2\langle\sigma_1^2\rangle}\right) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}, \quad (2.9)$$

where  $2\sigma = g_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu)$ . In the case of  $2\sigma_0 = \eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu) = 0$ , due to the quantum metric fluctuation ( $h_{\mu\nu}$ ), we have  $\langle\sigma_1^2\rangle \neq 0$  so we get

$$\langle G_T(0) \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle\sigma_1^2\rangle}} + \frac{m^2}{8\pi} \frac{1}{2}. \quad (2.10)$$

It should be noted that  $\langle\sigma_1^2\rangle$  is related to the density of gravitons [8].

By using the Fourier transformation of Dirac delta function,

$$-\frac{1}{8\pi} \delta(\sigma_0) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-x')} \mathcal{P}\mathcal{P} \frac{1}{p^2},$$

or equivalently

$$\frac{1}{8\pi^2} \frac{1}{\sigma_0} = - \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-x')} \pi \delta(p^2),$$

for the second part of Green function, we obtain

$$\frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}} = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-x')} \mathcal{P}\mathcal{P} \frac{m^2}{p^2(p^2 - m^2)}. \quad (2.11)$$

And for the first part we have

$$-\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle\sigma_1^2\rangle}} \exp\left[-\frac{(x-x')^4}{4\langle\sigma_1^2\rangle}\right] = \int \frac{d^4p}{(2\pi)^4} e^{-ik \cdot (x-x')} \tilde{G}_1(p).$$

Therefore, we obtain

$$\langle \tilde{G}_T(p) \rangle = \tilde{G}_1(p) + \mathcal{P}\mathcal{P} \frac{m^2}{p^2(p^2 - m^2)}. \quad (2.12)$$

In the previous paper, we proved that in the one-loop approximation, the Green function in Krein space quantization which appears in the transition amplitude is [7]:

$$\langle \tilde{G}_T(p) \rangle|_{\text{one-loop}} \equiv \tilde{G}_T(p)|_{\text{one-loop}} \equiv \mathcal{P}\mathcal{P} \frac{m^2}{p^2(p^2 - m^2)}. \quad (2.13)$$

That means in the one loop approximation, the contribution of delta function is negligible. It is worth to mention that in order to improve the UV behavior in relativistic higher-derivative correction theories, the propagator (2.13) has been used by some authors [12, 13]. It is also appear in supersymmetry (equation (20.76) in [14]).

### 3 Scalar effective potential

The effective action in the one-loop approximation for  $\lambda\phi^4$  scalar field is defined by [11, 10]

$$\begin{aligned}\Gamma(\phi) &= I(\phi) + \frac{i}{2}\hbar \text{Tr} \ln [1 + (\square + m^2)^{-1}V''(\phi)] + O(\hbar^2) \\ &= I(\phi) + \frac{i}{2}\hbar \text{Tr} \ln [1 - G_F V''(\phi)] + O(\hbar^2),\end{aligned}\tag{3.14}$$

where  $G_F$  is the Feynman Green function and

$$\text{Tr} \ln [1 - G_F V''(\phi)] = \int d^4x \langle x | \ln [1 - G_F V''(\phi)] | x \rangle.$$

By using the Fourier transformation, one obtains

$$\text{Tr} \ln [1 - G_F V''(\phi)] = \int d^4x \int \frac{d^4p}{(2\pi)^4} \ln \left[ 1 - V''(\phi) \left( \frac{1}{p^2 - m^2 + i\epsilon} \right) \right].$$

The effective potential is

$$V_{eff} = V_{eff}^{(0)} + \hbar V_{eff}^{(1)} + \hbar^2 V_{eff}^{(2)} + \dots,$$

where

$$\begin{aligned}V_{eff}^{(0)} &= \frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!}, \\ V_{eff}^{(1)} &= -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \ln \left[ 1 - \frac{V''(\phi)}{p^2 - m^2 + i\epsilon} \right].\end{aligned}\tag{3.15}$$

There are two different types of singularity in the one-loop effective potential [10]

$$\begin{aligned}V_{eff}^{(1)} &= \frac{1}{32\pi^2} \left( -\frac{m^2\lambda\phi^2}{2} \Gamma(-1) + \frac{1}{2} \left( \frac{\lambda\phi^2}{2} \right)^2 \Gamma(0) \right) \\ &+ \frac{1}{64\pi^2} \left[ \left( \frac{\lambda\phi^2}{2} + m^2 \right)^2 \ln \left( 1 + \frac{\lambda\phi^2}{2m^2} \right) - \frac{\lambda\phi^2}{2} \left( m^2 + \frac{3\lambda\phi^2}{4} \right) \right].\end{aligned}\tag{3.16}$$

By using the Green function (2.13), we have

$$V_{eff}^{(1)} = \frac{-i}{2} \int \frac{d^4p}{(2\pi)^4} \ln \left[ 1 - V''(\phi) \mathcal{P} \mathcal{P} \left( \frac{m^2}{p^2(p^2 - m^2)} \right) \right],\tag{3.17}$$

after some calculations (Appendix), we reach the following form for the effective action in the one loop approximation:

$$V_{eff}^{(1)} = -\frac{m^2\lambda\phi^2}{(16\pi)^2} \left[ \left( 1 + \frac{\lambda\phi^2}{m^2} \right) \left( -2 \ln \left( 1 + \frac{\lambda\phi^2}{m^2} \right) \right) + \ln \left( 1 + \frac{\lambda\phi^2}{2m^2} \right) + \ln \frac{\lambda\phi^2}{2m^2} + 2 \ln 2 \right]$$

$$+ \ln\left(1 + \frac{\lambda\phi^2}{2m^2}\right) - \ln\frac{\lambda\phi^2}{2m^2} \Big] - \frac{m^2\lambda\phi^2}{64\pi} \left[ 1 + \sqrt{1 + \frac{2m^2}{\lambda\phi^2}} - 2\sqrt{1 + \frac{m^2}{\lambda\phi^2}} \right]. \quad (3.18)$$

So, it is clear that in this method, the effective potential does not contain any divergence. In other words, in the Krein space method, the effective action is automatically regularized.

Now let's consider the effective mass and coupling constant. They are defined by

$$m_{eff}^2(\mu) = \left. \frac{d^2 V_{eff}}{d\phi^2} \right|_{\phi=\mu},$$

$$\lambda_{eff}(\mu) = \left. \frac{d^4 V_{eff}}{d\phi^4} \right|_{\phi=\mu}.$$

For our effective potential in  $\mu \rightarrow 0$ , we obtain

$$m_{eff}^2 = m^2 \left( 1 - \frac{\lambda}{32\pi} - \frac{\lambda}{64\pi^2} \ln 2 \right) + O(\lambda^2),$$

where  $m_{eff}$  and  $m$  are measurable quantities, they can be interpreted as the physical particle interaction mass and the physical free particle mass.

$\lambda_{eff}$  is a coupling constant in the presence of interaction which is finite. It is a function of a constant  $\mu$  which is the energy scale of the interaction,

$$\lambda_{eff} \equiv \lambda_\mu = \lambda - \frac{\lambda^2}{(8\pi)^2} \left[ 6 \ln \frac{\mu^2}{m^2} + 19 + 12 \ln 2 \right] + O(\lambda^3).$$

By defining  $\mu = e^{-t}$  and the running coupling constant  $\bar{\lambda}(t, \lambda)$ , the Beta function is

$$\beta = \frac{d\bar{\lambda}(t, \lambda)}{dt} = \frac{3\lambda^2}{16\pi^2}. \quad (3.19)$$

Our potential is different from previous methods but the interesting point is that in the one-loop approximation the  $\beta$  function does not change.

## 4 Conclusion and outlook

We recall that the negative frequency solutions of the field equation will be needed for quantizing in a correct way like the minimally coupled scalar field in de Sitter space. Contrary to the Minkowski space, the elimination of de Sitter negative norm in the minimally coupled states breaks the de Sitter invariance. Then, for restoring the de Sitter invariance, one needs to take into account the negative norm states *i.e.* the Krein space quantization. It provides a natural tool for eliminating the singularity in the QFT [5].

A theory becomes sensible when it can explain the experimental data for physical quantities and moreover its capacity for predicting the amount of new quantities which have not been measured yet. Every new observed quantity that is in agreement with what the theory predicts is a support for the theory.

In Quantum Field Theory, calculation of quantum effects of the physical quantities is made by the means of the expectation values which are related to the Green's function. Since the divergences of Green's functions usually appear in one of the following formats:

$$\lim_{\sigma \rightarrow 0} \frac{1}{\sigma}, \quad \ln \sigma, \quad \delta(\sigma),$$

so the divergence is automatically involved the calculations. It has been observed that in the Krein space quantization, by considering metric fluctuations, the divergence of the Green's function is removed. Therefore this method of quantization can be regarded as a new method for regularization, (Krein regularization).

After this regularization, one can proceed renormalization according to the previous method. This new kind of regularization may be utilized in the calculation of the Lamb-Shift and Magnetic-Anomaly. If we consider the renormalization point at  $p^2 = m^2$ , we will exactly obtain the previous reported results for the Lamb-Shift and Magnetic-Anomaly in the one-loop approximation [15].

In this paper, Krein space quantization has been used to calculate the effective action for  $\lambda\phi^4$  theory in Minkowski space-time in the one-loop approximation. It is found that in this approximation the theory is free of any divergence since the Green function is free of any divergence in the ultraviolet and infrared limit. Our potential is different from previous methods but in the one-loop approximation, the  $\beta$  function does not change. In this approximation and for scalar field, we see that this quantization eliminates the singularity in the theory without changing the physical content of the theory. As a future work, one may use this method for considering the quantum gravity in the background field method. This method may solved the non-renormalizability of quantum gravity in the background field method.

**Acknowledgments:** The author would like to thank M. R. Tanhayi.

## A Appendix

In this appendix, we explicitly calculate the integral (3.17). We have

$$\begin{aligned} V_{eff}^{(1)} &= \frac{-i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left[ 1 - V''(\phi) \mathcal{P} \mathcal{P} \left( \frac{m^2}{p^2(p^2 - m^2)} \right) \right] = \\ &= \frac{-i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left[ 1 - \frac{m^2 V''(\phi)}{2} \left( \frac{1}{p^2(p^2 - m^2) + i\epsilon} + \frac{1}{p^2(p^2 - m^2) - i\epsilon} \right) \right] = \end{aligned}$$

$$\frac{-i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left[ \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) - i\epsilon} \right) \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) + i\epsilon} \right) - \left( \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2)} \right)^2 \right], \quad (\text{A.1})$$

where  $\epsilon^2$  has been vanished. One can write

$$\begin{aligned} \frac{-i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left[ \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) - i\epsilon} \right) \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) + i\epsilon} \right) \right. \\ \left. \left( 1 - \frac{\left( \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2)} \right)^2}{\left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) - i\epsilon} \right) \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) + i\epsilon} \right)} \right) \right]. \quad (\text{A.2}) \end{aligned}$$

So we have

$$\begin{aligned} \frac{-i}{2} \int \frac{d^4 p}{(2\pi)^4} \left[ \ln \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) - i\epsilon} \right) + \frac{-i}{2} \ln \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) + i\epsilon} \right) \right] + \\ \frac{-i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left[ 1 - \left( \frac{\left( \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2)} \right)^2}{\left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) - i\epsilon} \right) \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) + i\epsilon} \right)} \right) \right]. \quad (\text{A.3}) \end{aligned}$$

After some calculations we obtain

$$\begin{aligned} \frac{-i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) - i\epsilon} \right) + \frac{-i}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) + i\epsilon} \right) + \\ \frac{-i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left[ 1 - \left( \frac{\frac{m^2 \lambda \phi^2}{4}}{p^2(p^2 - m^2) - \frac{m^2 \lambda \phi^2}{4}} \right)^2 \right]. \quad (\text{A.4}) \end{aligned}$$

By using Wick rotation ( $p_0 \rightarrow ik_0$ ), the corresponding Euclidean four-momentum  $k$ , we find that the two first integrals are vanished and the third splits into two integrals

$$\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{k^2(k^2 + m^2) - \frac{m^2 \lambda \phi^2}{4}} \right) + \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left( 1 + \frac{\frac{m^2 \lambda \phi^2}{4}}{k^2(k^2 + m^2) - \frac{m^2 \lambda \phi^2}{4}} \right). \quad (\text{A.5})$$

Now we want to solve these integrals. The first term is:

$$\begin{aligned} \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{k^2(k^2 + m^2) - \frac{m^2 \lambda \phi^2}{4}} \right) &= \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left( 1 - \frac{\frac{m^2 \lambda \phi^2}{4}}{(k^2 + \frac{m^2}{2})^2 - \frac{m^4}{4} - \frac{m^2 \lambda \phi^2}{4}} \right) = \\ &= -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^{\frac{m^2 \lambda \phi^2}{4}} \frac{du}{(k^2 + \frac{m^2}{2})^2 - \frac{m^4}{4} - \frac{m^2 \lambda \phi^2}{4} - u} = \end{aligned}$$



$$-\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^{\frac{m^2 \lambda \phi^2}{4}} du \int_0^\infty dt \exp \left[ -t \left( q^2 - \frac{m^4}{4} - \frac{m^2 \lambda \phi^2}{4} - u \right) \right], \quad (\text{A.6})$$

where  $q = k^2 + \frac{m^2}{2}$  and it is necessary  $q^2 > \frac{m^4}{4} + \frac{m^2 \lambda \phi^2}{2}$ . So, we see that there is a cutoff momentum, after doing some straightforward calculations one obtains  $k_c = \sqrt{\frac{\lambda \phi^2}{2}}$  or  $k_c < k < \infty$ . We obtain

$$\begin{aligned} & -\frac{1}{32\pi^2} \int_0^\infty dt \left( \int_{q_c}^\infty dq \left( q - \frac{m^2}{2} \right) e^{-tq^2} \int_0^{\frac{m^2 \lambda \phi^2}{4}} du e^{tu} \right) \exp \left[ t \left( \frac{m^4}{4} + \frac{m^2 \lambda \phi^2}{4} \right) \right] = \\ & -\frac{1}{32\pi^2} \int_0^\infty dt \left( \frac{1}{2t} e^{-tq_c^2} - \frac{m^2}{4} \sqrt{\frac{\pi}{t}} e^{-tq_c^2} \right) \left( \frac{1}{t} (e^{\frac{1}{4}tm^2\lambda\phi^2} - 1) \right) \exp \left[ t \left( \frac{m^4}{4} + \frac{m^2 \lambda \phi^2}{4} \right) \right] \equiv A \end{aligned}$$

The second term in eq. (A.5) is

$$\begin{aligned} & \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left( 1 + \frac{\frac{m^2 \lambda \phi^2}{4}}{k^2(k^2 + m^2) - \frac{m^2 \lambda \phi^2}{4}} \right) = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left( 1 + \frac{\frac{m^2 \lambda \phi^2}{4}}{(k^2 + \frac{m^2}{2})^2 - \frac{m^4}{4} - \frac{m^2 \lambda \phi^2}{4}} \right) = \\ & \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^{\frac{m^2 \lambda \phi^2}{4}} \frac{du}{(k^2 + \frac{m^2}{2})^2 - \frac{m^4}{4} - \frac{m^2 \lambda \phi^2}{4} + u} = \\ & \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^{\frac{m^2 \lambda \phi^2}{4}} du \int_0^\infty dt \exp \left[ -t \left( q^2 - \frac{m^4}{4} - \frac{m^2 \lambda \phi^2}{4} + u \right) \right] \quad (\text{A.7}) \\ & \frac{1}{32\pi^2} \int_0^\infty dt \left( \int_{q_c}^\infty dq \left( q - \frac{m^2}{2} \right) e^{-tq^2} \int_0^{\frac{m^2 \lambda \phi^2}{4}} du e^{-tu} \right) \exp \left[ t \left( \frac{m^4}{4} + \frac{m^2 \lambda \phi^2}{4} \right) \right] = \\ & \frac{1}{32\pi^2} \int_0^\infty dt \left( \frac{1}{2t} e^{-tq_c^2} - \frac{m^2}{4} \sqrt{\frac{\pi}{t}} e^{-tq_c^2} \right) \left( -\frac{1}{t} (e^{-\frac{1}{4}tm^2\lambda\phi^2} - 1) \right) \exp \left[ t \left( \frac{m^4}{4} + \frac{m^2 \lambda \phi^2}{4} \right) \right] \equiv B \end{aligned}$$

By summing A and B we get

$$\begin{aligned} A + B &= -\frac{1}{32\pi^2} \sum_{n=1}^\infty \int_0^\infty \frac{dt}{4^{2n}(2n)!} t^{2n-2} (m^2 \lambda \phi^2)^{2n} \exp \left[ -t \frac{\lambda \phi^2}{4} (m^2 + \lambda \phi^2) \right] + \\ & \frac{m^2}{64\pi^{\frac{3}{2}}} \sum_{n=1}^\infty \int_0^\infty \frac{dt}{4^{2n}(2n)!} t^{2n-\frac{3}{2}} (m^2 \lambda \phi^2)^{2n} \exp \left[ -t \frac{\lambda \phi^2}{4} (m^2 + \lambda \phi^2) \right]. \quad (\text{A.8}) \end{aligned}$$

Then for the one loop case we have

$$V_{eff}^{(1)} = -\frac{1}{32\pi^2} \sum_{n=1}^\infty \int_0^\infty \frac{dt}{4^{2n}(2n)!} t^{2n-2} (m^2 \lambda \phi^2)^{2n} \exp \left[ -t \frac{\lambda \phi^2}{4} (m^2 + \lambda \phi^2) \right] +$$

$$\frac{m^2}{64\pi^{\frac{3}{2}}} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dt}{4^{2n}(2n)!} t^{2n-\frac{3}{2}} (m^2 \lambda \phi^2)^{2n} \exp \left[ -t \frac{\lambda \phi^2}{4} (m^2 + \lambda \phi^2) \right], \quad (\text{A.9})$$

or

$$V_{eff}^{(1)} = -\frac{1}{(16\pi)^2} \sum_{n=1}^{\infty} \frac{m^2 \lambda \phi^2}{n(2n-1)} \frac{1}{(1 + \frac{\lambda \phi^2}{m^2})^{2n-1}} + \frac{m^2}{64\pi^{\frac{3}{2}}} \sum_{n=1}^{\infty} \sqrt{m^2 \lambda \phi^2} \frac{(2n - \frac{3}{2})!}{(2n)!} \frac{1}{(1 + \frac{\lambda \phi^2}{m^2})^{2n-\frac{1}{2}}}. \quad (\text{A.10})$$

And finally we have

$$V_{eff}^{(1)} = -\frac{m^2 \lambda \phi^2}{(16\pi)^2} \left[ \left( 1 + \frac{\lambda \phi^2}{m^2} \right) \ln \left( 1 - \frac{1}{(1 + \frac{\lambda \phi^2}{m^2})^2} \right) + \ln \left( \frac{2 + \frac{\lambda \phi^2}{m^2}}{\frac{\lambda \phi^2}{m^2}} \right) \right] - \frac{m^2 \lambda \phi^2}{64\pi} \left[ 1 + \sqrt{1 + \frac{2m^2}{\lambda \phi^2}} - 2\sqrt{1 + \frac{m^2}{\lambda \phi^2}} \right], \quad (\text{A.11})$$

or

$$V_{eff}^{(1)} = -\frac{m^2 \lambda \phi^2}{(16\pi)^2} \left[ \left( 1 + \frac{\lambda \phi^2}{m^2} \right) \left( -2 \ln(1 + \frac{\lambda \phi^2}{m^2}) + \ln(1 + \frac{\lambda \phi^2}{2m^2}) + \ln \frac{\lambda \phi^2}{2m^2} + 2 \ln 2 \right) + \ln(1 + \frac{\lambda \phi^2}{2m^2}) - \ln \frac{\lambda \phi^2}{2m^2} \right] - \frac{m^2 \lambda \phi^2}{64\pi} \left[ 1 + \sqrt{1 + \frac{2m^2}{\lambda \phi^2}} - 2\sqrt{1 + \frac{m^2}{\lambda \phi^2}} \right]. \quad (\text{A.12})$$

## References

- [1] I. Antoniadis, J. Iliopoulos, T.N. Tomaras, Nuclear Phys. B, 462(1996)437.
- [2] T. Garidi et al, J. Math. Phys., 49(2008)032501; T. Garidi et al, J. Math. Phys., 44(2003)3838; S. Behroozi et al, Phys. Rev. D, 74(2006)124014.
- [3] M. Dehghani et al, Phys. Rev. D, 77(2008)064028; M.V. Takook et al, J. Math Phys., 51(2010)032503.
- [4] B. Allen, Phys. Rev. D, 32(1985)3136.
- [5] J.P. Gazeau, J. Renaud, M.V. Takook, Class. Quan. Grav., 17(2000)1415, gr-qc/9904023.
- [6] M.V. Takook, Mod. Phys. Lett. A, 16(2001)1691, gr-qc/0005020.
- [7] M.V. Takook, Int. J. Mod. Phys. E, 11(2002)509, gr-qc/0006019.
- [8] H.L. Ford, Quantum Field Theory in Curved Spacetime, gr-qc/9707062.

- [9] S. Rouhani, M.V. Takook, Int. J. Theor. Phys., 48(2009)27402747.
- [10] C. Itzykson, J.B. Zuber, McGraw-Hill, Inc. (1988) *Quantum Field Theory*.
- [11] N.D. Birrell, P.C.W. Davies, Cambridge University Press, (1982) *QUANTUM FIELD IN CURVED SPACE*.
- [12] N.H. Barth, S.M. Christensen, Phys. Rev. D, 28(1983)1876.
- [13] P. Horava, Phys. Rev. D, 79(2009)084008, arXiv:0901.3775.
- [14] M. Kaku, Oxford University Press, (1993) *Quantum Field Theory*.
- [15] A. Zarei, B. Forghan, M.V. Takook, *QED in Krein Regularization*, In preparation (2010).